Interferometry on a blazed grating *

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Abstract

Interferometric calculations are done on a flat blazed grating. It is found that to maximize the efficiency,

- Order of diffraction m must be as small as possible.
- Incident angle α and emergent angle β must be as close as possible.
- Groove interval d must be as big as possible.

to avoid shadowing. These parameters will be decided from the physical limitation of production of the grating and assembly of the spectrometer.

Secondary peaks beyond the first local minima has 4.7% peak intensity of the primary peak. This is not considered in efficiency calculation.

For the K-band, calculated efficiency is more than about 96% throughout the band when $\alpha = \beta$, $m = \pm 1$, and $d \sim 22\mu m$. With larger d, efficiency increases. When α and β are separated, efficiency drops to about 95% for 20 deg difference.

When the slit width is wider than a certain limit, namely, spatial resolution of the instrument is less than the diffraction limit of the telescope, spectral resolution of the spectrometer is limited by the geometrical size of the slit. When the field of view (FOV) of a spatial element is n^2 of diffraction limited spatial elements, we need about n times more grating grooves to achieve the same spectral resolving power for a diffraction limited instrument.

1 Interferometric calculation

Interferometry is calculated on a grating with groove interval d and groove angle ε as in Figure 1.

1.1 Geometry

Incident light from $P_1(x_1, y_1, 0)$ with signed angle α comes from distance r_1'. Consider detecting emergent light with signed angle β at $P_2(x_2, y_2, 0)$ distance

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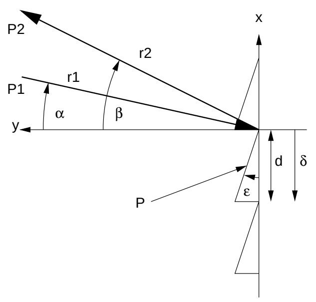


Figure 1: Geometry of light and blazed grating

r_2'. Here, light reflected at the point $P(x, y, 0)$ is considered. Throughout this report, refractive index of media around the grating is assumed to be 1.

Distance between P_1 and P is $r_1^2 = (x_1 - x)^2 + (y_1 - y)^2$ and distance between P_1 and the coordinate origin is $r_1'^2 = x^2 + y^2$. So, r_1 can be approximated as $r_1 \sim r_1' - (x_1 x - y_1 y) \div r_1'$. Likewise $r_2 \sim r_2' - (x_2 x - y_2 y) \div r_2'$. Using the incident and emergent angles, $x_1 = r_1' \sin\alpha$ and $y_1 = r_1' \sin\beta$. Therefore, light path going through P can be explained as $r_1 \sim r_1' - (x \sin\alpha + y \cos\alpha)$ and $r_2 \sim r_2' - (x \sin\beta + y \cos\beta)$.

On the nth groove of total N grooves, $x = (n - \frac{N}{2})d - \delta$ and $y = \delta \tan\varepsilon$. Therefore, phase delay $\phi = k \times (r_1 + r_2)$ at P_2 of the light from P_1 and reflected at P is

$$
\begin{aligned}
&\frac{\phi(\delta, n)}{k} \\
&\sim \quad r_1' + r_2' - [\{(n - \frac{N}{2})d - \delta\}(\sin\alpha + \sin\beta) + \delta \tan\varepsilon(\cos\alpha + \cos\beta)] \quad (1) \\
&= \quad r_1' + r_2' + \frac{N}{2}d(\sin\alpha + \sin\beta) \\
&\qquad -nd(\sin\alpha + \sin\beta) + \delta \frac{1}{\cos\varepsilon}(\sin[\alpha - \varepsilon] + \sin[\beta - \varepsilon]) \quad (2)
\end{aligned}
$$

where $k = 2\pi/\lambda$. Here, we define

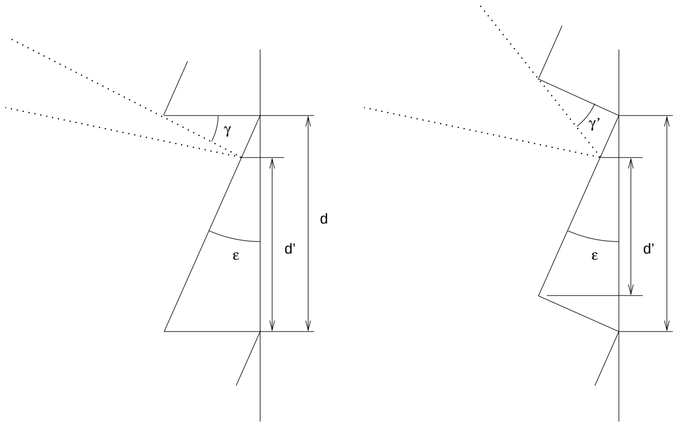
$$A = \sin\alpha + \sin\beta \tag{3}$$

and

$$E = \frac{1}{\cos\varepsilon}(\sin[\alpha - \varepsilon] + \sin[\beta - \varepsilon]) \tag{4}$$

1.2 Integration

Using the equations, complex amplitude of the light detected at P_2 is calculated from integration over the grating surface. Here we take the shadowing effect



(a) (b)

Figure 2: Shadowing of the incident or emergent light on different grating shapes.

into account. As shown in Figure 2, only d' of the groove surface reflects the light.

$$U(P_2) \propto \sum_{n=0}^{N} \int_0^{d'} \exp[i\phi(\delta, n)] d\delta \tag{5}$$

$$\propto \sum_{n=0}^{N} \exp[-ikndA] \times \int_0^{d'} \exp[ik\delta E] d\delta \tag{6}$$

$$\propto \frac{1 - \exp[-ikNdA]}{1 - \exp[-ikdA]} \times \frac{1}{ikE}(\exp[ikd'E] - 1) \tag{7}$$

Normalizing at $A = 0$ and $E = 0$, intensity of the light detected at P_2 is

$$I(P_2) = \frac{1}{N^2} \frac{1 - \cos[kNdA]}{1 - \cos[kdA]} \times \frac{2}{(kd'E)^2}(1 - \cos[kd'E]) \tag{8}$$

1.3 Grating function

The first term of Equation 8, which we call grating function is from the periodical sum over the grooves of the grating which has peaks at $kdA = 2\pi m$ so

$$m\lambda = d(\sin\alpha + \sin\beta) \tag{9}$$

When the incident angle α and diffraction order m is known, emergent angle β can be calculated as

$$\beta = \arcsin[\frac{m\lambda}{d} - \sin\alpha] \tag{10}$$

where m is an integer of the diffraction order.

On the other hand, when we define difference of incident and emergent angles,

$$\Delta \equiv \alpha - \beta \tag{11}$$

α and β is calculated as follows with fixed m:

$$\alpha,\beta = \arcsin\left[\frac{\frac{m\lambda}{d}(1+\cos\Delta) \pm \sin\Delta \times \sqrt{2(1+\cos\Delta) - (\frac{m\lambda}{d})^2}}{2(1+\cos\Delta)}\right] \tag{12}$$

Derivation is described in Appendix A

1.4 Slit function

The second term of Equation 8, which is called slit function, is the diffraction from the grooves. This peaks at $E = 0$ or

$$\alpha + \beta = 2\varepsilon \tag{13}$$

Distance from the peak to the first minima is almost at the same interval that the first term has for the peaks on A.

2 Spectrometer with a narrow entrance slit

When the entrance slit is narrow enough, Equation 8 can be used to asses properties of the output spectra. As shown below, the slit can be treated narrow and $I(P_2)$ can be treated as instrumental profile when difference of incident light direction (half width) $\delta\alpha$ is smaller than $\sim \lambda/(2Nd\cos[\alpha_0])$ where α_0 is the center direction of the incident light. In Figure 3, left-most curve shows normalized profile of $I(P_2)$ at $kdA \sim 0$ and $kd'E \sim 0$. Horizontal axis is change of $kNd\sin\beta$ from the center of the spectra.

2.1 Spectral resolution

For $N \gg 1$, spectral resolution of the grating is calculated as follows. First, we have a peak at $A_1 = \lambda_1 m/d$ for a wavelength λ_1. Near the peak, there are local minima at $A_2 = A_1 \pm \lambda_1/(Nd)$. We define spectral resolution when the next spectral element of λ_2 is at the local minima A_2. Namely, for the mth order, we have a peak for λ_2 at $A_2 = \lambda_2 m/d$. Thus, the wavelength difference $\Delta\lambda = |\lambda_2 - \lambda_1|$ or

$$\Delta\lambda = \lambda/(mN) \tag{14}$$

or

$$R \equiv \frac{\lambda}{\Delta\lambda} = mN \tag{15}$$

2.2 Secondary peaks

Beyond the local minima, there are second local maxima near A' that makes $\cos[kdNA'] = -1$ beyond the first local minima. For the peak of mth order diffraction of λ, $A' = \lambda/d \times (m + 3/N)$. The position A' corresponds to the peaks for $\lambda' = \lambda + 3\Delta\lambda$.

Near the secondary peaks, from $kdA = 2\pi m$, grating function is

$$
\begin{aligned}
I &\sim \frac{1}{N^2}\frac{1-\cos[kdNA']}{1-\cos[kdA']} && (16)\\
&= \frac{1}{N^2}\frac{2}{1-\cos[3\pi/N]} && (17)\\
&\sim \frac{1}{N^2}\frac{2}{1/2(3\pi/N)^2} && (18)\\
&\sim 4.5\% && (19)
\end{aligned}
$$

assuming $N \gg 3\pi$.

Exact positions of secondary peaks can be obtained by taking differential of the grating function. Position and value of the peaks weakly depends on N. The secondary peaks are nearer to the primary peak and has intensity about 4.7 % of the primary peak when N is more than about 1000.

3 Spectrometer with a wide entrance slit

3.1 Geometrical limit of spectral profile

When the entrance slit is wide, Equation 8 has to be integrated over the entrance slit, assuming that the light coming to different locations on the entrance slit is not coherent. This results in spectral resolution not being geometrically limited rather than being diffraction limited as for a narrow entrance slit.

When the direction of the collimated light spreads from $\alpha_1 \equiv \alpha_0 - \delta\alpha$ to $\alpha_2 \equiv \alpha_0 + \delta\alpha$, Equation 8 has to be integrated from α_1 to α_2 and profile of the spectra can be assessed.

$$
\begin{aligned}
J(P_2) &\equiv \int_{\alpha_1}^{\alpha_2} I(P_2)\ \mathrm{d}\alpha && (20)\\
&= \int_{A_1}^{A_2} I(P_2)\cos\alpha\ \mathrm{d}A && (21)\\
&= \int_{A_1}^{A_2} \frac{1}{N^2}\frac{1-\cos[kNdA]}{1-\cos[kdA]}\frac{2}{(kd'E)^2}(1-\cos[kd'E])\cos\alpha\ \mathrm{d}A && (22)\\
& && (23)
\end{aligned}
$$

with $A_{1,2} \equiv \sin\alpha_{1,2} + \sin\beta$. We can assume $\delta\alpha \ll 1$ (slit width is small, if not narrow) and $kd'E \ll 1$ (the spectrometer is used around the Bragg's condition).

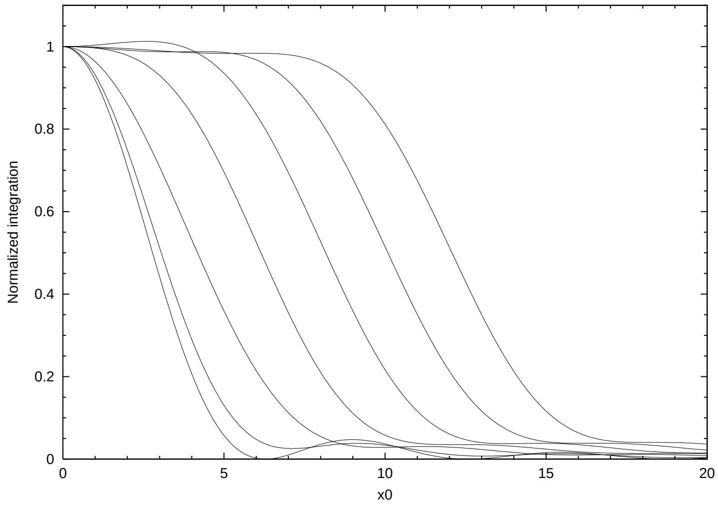


Figure 3: Numerical integration of Equation 26. From left to right, curves show results for δx of 0, 2, 4, 6, 8, 10, and 12. Results are normalized at $x_0 = 0$.

In the case, $\cos\alpha \sim \cos\alpha_0$ and $1 - \cos[kd'E] \sim (kd'E)^2/2$. When the grating is used in the mth order, A can be replaced with $A' \equiv A - 2\pi m/(kd)$. Near the α and β of spectra we are observing, $A' \ll 1$. Therefore, $1 - \cos[kdA] = 1 - \cos[kdA'] \sim (kdA)^2/2$. Here, we define $A'_{1,2} = A_{1,2} - 2\pi m/(kd) \sim \sin\alpha_0 + \sin\beta - 2\pi m/kd \pm \cos[\alpha_0]\delta\alpha$ and the integration becomes

$$
\begin{aligned}
J(P_2) &\sim \cos\alpha_0 \int_{A'_1}^{A'_2} \frac{2(1 - \cos[kNdA'])}{(kNdA')^2}\,\mathrm{d}A' && (24)\\
&= \frac{\cos\alpha_0}{kNd} \int_{x_0-\delta x}^{x_0+\delta x} \frac{2(1-\cos x)}{x^2}\,\mathrm{d}x && (25)\\
&\propto \int_{x_0-\delta x}^{x_0+\delta x} \frac{1-\cos x}{x^2}\,\mathrm{d}x && (26)
\end{aligned}
$$

with $x_0 \equiv kNd \times (\sin\alpha_0 + \sin\beta - 2\pi m/kd)$ and $\delta x \equiv kNd \times \cos[\alpha_0]\delta\alpha$. The equation cannot be integrated analytically.

Figure 3 shows results from numerical integration of Equation 26 with different δx. For δx smaller than $\sim \pi$, width of the spectra is comparable with that for $\delta x = 0$. On the other hand, for $\delta x > \pi$, the profile gets half of the center at $x_0 \sim \delta x$.

From the definition, $\delta x \sim \pi$ corresponds to a threshold angular half width of the slit

$$\delta\alpha_{th} \sim \pi/(kNd\cos[\alpha_0]) = \lambda/(2Nd\cos[\alpha_0]). \quad (27)$$

If the half width of the direction of the incident light onto the grating is more than $\delta\alpha_{th}$, spectral resolution of the spectrometer is not limited by diffraction of the grating, but limited by geometrical width of the slit.

3.2 Relation with system $A\Omega$

When the instrument is spatially diffraction limited at λ_0, field of view (FOV) of a spatial element on a telescope with entrance pupil diameter of D_{tel} is in the order of λ_0/D_{tel}. Thus, the system $A\Omega$ becomes $\sim \lambda_0^2\pi^2/16$. On a seeing limited instrument, FOV of a spatial element is more than for the diffraction limited instrument. Here, we assume to observe FOV n^2 times larger than that for a diffraction limited instrument. In the case, system $A\Omega$ becomes $\sim n^2\lambda_0^2\pi^2/16$.

When the diameter of the grating is D_{gr}, beam width of incident light becomes $D_{gr}\cos[\alpha_0]$. Considering only about the beam in the dispersion direction, solid angle of the light going into the grating is $\Omega_{gr} \sim n^2\lambda_0^2\pi/(4D_{gr}^2\cos[\alpha_0])$. Therefore, from $D_{gr} = Nd$, half width of the incident beam direction is

$$\delta\alpha \sim n\lambda_0/(2D_{gr}\cos[\alpha_0]) \sim n\delta\alpha_{th}. \tag{28}$$

For a seeing limited instrument, spectral resolution is limited by the geometrical image size of the entrance slit, rather than the diffraction of the grating.

3.3 Spectral resolution

When the spectral resolution of the spectrometer is geometrically limited, spectral resolving power is calculated as follows. Half width $\delta\beta$ of the slit image for a certain wavelength in the emergent light from the grating is

$$\begin{aligned} \delta\beta &= \left|\frac{\mathrm{d}\beta}{\mathrm{d}\alpha}(A = const.)\right| \times \delta\alpha & (29)\\ &\sim \frac{\cos[\alpha_0]}{\cos\beta} \times \delta\alpha & (30)\end{aligned}$$

Therefore, from Equation 9, wavelength difference becomes

$$\begin{aligned} \Delta\lambda &= \frac{d\cos\beta}{m} \times 2\delta\beta & (31)\\ &\sim \frac{2d\cos[\alpha_0]\delta\alpha}{m} & (32)\end{aligned}$$

where $\Delta\lambda$ is the full width. Therefore spectral resolving power becomes

$$\begin{aligned} R &\equiv \frac{\lambda}{\Delta\lambda} & (33)\\ &\sim \frac{m\lambda}{2d\cos[\alpha_0]\delta\alpha} & (34)\\ &\sim \frac{mD_{gr}}{nd} \times \frac{\lambda}{\lambda_0} & (35)\\ &\sim \frac{mN}{n} \times \frac{\lambda}{\lambda_0} & (36)\end{aligned}$$

Comparing this with Equation 15, resolving power of the spectrometer is degraded by a factor of $1/n\cos[\alpha_0]$ for $\lambda \sim \lambda_0$. As a consequence, we need $n\cos[\alpha_0]$ times more grooves for a seeing limited spectrometer to achieve the same spectral resolving power as for a diffraction limited spectrometer.

3.4 $A\Omega$ consideration

Spectral resolution of the spectrometer is considered from conservation of $A\Omega$. With telescope aperture diameter D_{tel}, width of the slit on the sky θ_{tel}, square root of system $A\Omega$ is $D_{tel} \times \theta_{tel}$.

Grating width defined from system $A\Omega$ and camera focal length On the focal plane array (FPA), image of the slit has to be D_{img}. Thus, with width of the light direction conversing to the FPA θ_{img}, focal ratio of the spectrometer camera is

$$F_{cam} \equiv \frac{1}{2\tan(\theta_{img}/2)} \tag{37}$$

$$\sim \frac{1}{\theta_{img}} \tag{38}$$

$$= \frac{D_{img}}{D_{tel}\theta_{tel}} \tag{39}$$

Therefore, with focal length of spectrometer camera f_{cam} diameter of beam emerging from the grating is

$$D_{out} = \frac{f_{cam}}{F_{cam}} = \frac{D_{tel}\theta_{tel}f_{cam}}{D_{img}} \tag{40}$$

and grating width is

$$D_{gr} = \frac{D_{out}}{\cos\beta} = \frac{D_{tel}\theta_{tel}f_{cam}}{D_{img}\cos\beta} \tag{41}$$

Camera focal length defined from spectral resolution Size of the image of the slit on the FPA is adjuste to be the same as the size of a spectral element dispersed by the grating. From Equation 9, width of the beam direction of a spectral element $\Delta\lambda$ is

$$\delta\beta = \frac{m\Delta\lambda}{d\cos\beta} \tag{42}$$

Therefore,

$$D_{img} = f_{cam} \times \delta\beta = \frac{f_{cam}m\Delta\lambda}{d\cos\beta} \tag{43}$$

Putting Equation 43 into Equation 41,

$$D_{gr} = \frac{D_{tel}\theta_{tel}d}{m\Delta\lambda} \tag{44}$$

and number of grating grooves illuminated by the beam is

$$N = \frac{D_{gr}}{d} = \frac{D_{tel}\theta_{tel}}{m\Delta\lambda} \tag{45}$$

Comparison with spectral and spatial resolution With spectral resolution $R \equiv \lambda/\Delta\lambda$, number of grooves becomes

$$N = \frac{D_{tel}\theta_{tel}R}{m\lambda} \tag{46}$$

Moreover, if slit FOV θ_{tel} is n times that of the diffraction limit of the telescope $\sim \lambda/D_{tel}$,

$$N = \frac{nR}{m} \tag{47}$$

4 Design of a seeing limited spectrograph

Sometimes, we need to adjust dispersion so that a spectral element is dispersed onto a disired size Δw, with the image size Δi of the slit being different. For this kind of requirements, design steps are considered here.

4.1 Requirements

First we define required spectral resolving power $R_{req} \equiv \lambda/\Delta/\lambda$ with $\Delta\lambda$ being full width of wavelenght for a spectral element. Moreover, groove interval d of the grating and diffraction order m, difference of incident angle and emergent angle Δ have to be defined. Also, pixel size of focal plane array and required spatial resolution defines spectral element Δw and spatial magnification of the spectrometer M.

Incidecnt angle α_0 and emergent angle β is calculated by Equation 12. Because overall size of the spectrometer is proportional to $\cos\beta$, the solution with larger $|\beta|$ is better than with larger $|\alpha_0|$.

4.2 Focal lengths

From Equation 9, wavelength dependency of dispersing direction is calculated as

$$\Delta\beta = \frac{m}{d\cos\beta}\Delta\lambda \tag{48}$$

With camera focal length f_{cam}, $\Delta w = f_{cam} \times \Delta\beta$. Therefore, camera focal length has to be

$$f_{cam} = \frac{Rd\cos\beta}{m\lambda} \times \Delta w. \tag{49}$$

From spatial magnification, focal length of the collimator becomes

$$f_{col} = \frac{1}{M} \times f_{cam}. \tag{50}$$

Naturally, width of the slit image is

$$\Delta i = \frac{\cos[\alpha_0]}{\cos\beta} M \Delta s \tag{51}$$

which can be calculated from Equation 9 with λ fixed.

4.3 Width of the grating and the beams

From focal length of the collimator, width of the grating is calculated as follows. When the FOV is n^2 times larger than that for a diffraction limited instrument, system $A\Omega$ becomes $\sim n^2\lambda_0^2\pi^2/16$. With the slit width Δs, light disperses with half angle of $n\lambda_0/(2\Delta s)$. Thus, beam width of the incident light would be

$$\begin{aligned} D_{in} &= \frac{n\lambda_0 f_{col}}{\Delta s} && (52)\\ &= Rn \times \frac{d\cos\beta}{m} \times \frac{\lambda_0}{\lambda} \times \frac{\Delta w}{M\Delta s} && (53) \end{aligned}$$

From geometry, grating width is

$$\begin{aligned} D_{gr} &= \frac{n\lambda_0 f_{col}}{\Delta s\cos[\alpha_0]} && (54)\\ &= Rn \times \frac{d}{m} \times \frac{\lambda_0}{\lambda} \times \frac{\Delta w\cos\beta}{M\Delta s\cos[\alpha_0]} && (55) \end{aligned}$$

And finally, emergent angle is

$$\begin{aligned} D_{out} &= \frac{n\lambda_0 f_{col}\cos\beta}{\Delta s\cos[\alpha_0]} && (56)\\ &= Rn \times \frac{d\cos\beta}{m} \times \frac{\lambda_0}{\lambda} \times \frac{\Delta w\cos\beta}{M\Delta s\cos[\alpha_0]} && (57) \end{aligned}$$

From D_{gr}, number of grating grooves $N = D_{gr}/d$ becomes

$$N = \frac{Rn}{m} \times \frac{\lambda_0}{\lambda} \times \frac{\Delta w\cos\beta}{M\Delta s\cos[\alpha_0]} \tag{58}$$

With slit image width Δi being $M\Delta s\cos[\alpha_0]/\cos\beta$, N is $\Delta w/\Delta i$ times more than that for simple seeing limited spectrographs.

In total, we have to have the grating width $n \times \Delta w/\Delta i$ times bigger than that of diffraction limited spectrographs.

5 Shadowing

5.1 Grating made with an acute angle cutting tool

This section considers the case (a) in Figure 2. When ε is not zero and α or β is not zero, some of the light does not reach the mirror surface of a groove.

When the light has the angle γ as in Figure 2 (a), d' can be calculated as

$$d' = d \times \frac{\cos\gamma \cos|\varepsilon|}{cos(\gamma - |\varepsilon|)} \tag{59}$$

where

$$\gamma = \begin{cases} 0 & (\alpha\varepsilon < 0 \text{ and } \beta\varepsilon < 0) \\ |\alpha| & (\alpha\varepsilon > 0 \text{ and } \beta\varepsilon \leq 0) \\ |\beta| & (\alpha\varepsilon \leq 0 \text{ and } \beta\varepsilon > 0) \\ max(|\alpha|, |\beta|) & (\alpha\varepsilon \geq 0 \text{ and } \beta\varepsilon \geq 0) \end{cases} \tag{60}$$

In a practical grating, we have to have $\varepsilon > 0$ and with β_c for the central wavelength λ_c of the spectrometer, $\alpha + \beta_c = 2\varepsilon$ to maximize the slit function. Within the requirements, d' is maximum when $\alpha = \beta_c$.

5.2 Grating made with a right angle cutting tool

For the case (b) in Figure 2, another geometry has to be considered. In this case,

$$d' = d \times \frac{\cos[|\varepsilon| - \gamma'] \cos|\varepsilon|}{cos\gamma'} \tag{61}$$

where

$$\gamma' = \begin{cases} 0 & ((\alpha - \varepsilon)\varepsilon < 0 \text{ and } (\beta - \varepsilon)\varepsilon < 0) \\ |\alpha| - |\varepsilon| & ((\alpha - \varepsilon)\varepsilon > 0 \text{ and } (\beta - \varepsilon)\varepsilon \leq 0) \\ |\beta| - |\varepsilon| & ((\alpha - \varepsilon)\varepsilon \leq 0 \text{ and } (\beta - \varepsilon)\varepsilon > 0) \\ max(|\alpha| - |\varepsilon|, |\beta| - |\varepsilon|) & ((\alpha - \varepsilon)\varepsilon \geq 0 \text{ and } (\beta - \varepsilon)\varepsilon \geq 0) \end{cases} \tag{62}$$

When γ is defined as in Equation 60, $\gamma' = \gamma - |\varepsilon|$ and Equations 61 and 62 are identical to Equations 59 and 60.

6 Splitting into other orders

The other sources for degradation of efficiency is light splitting into other orders of diffraction. Assuming the grating function as a delta function, intensity of the light in the mth order is calculated as follows.

First, defining the incident angle α, emergent angle β_m to the mth order is calculated from Equation 10. With

$$E_m = \frac{1}{\cos\varepsilon}(\sin[\alpha - \varepsilon] + \sin[\beta_m - \varepsilon]) \tag{63}$$

intensity of the emergent light in the mth order is

$$I_m = \frac{d'}{d}\frac{2}{(kd'E_m)^2}(1 - \cos[kd'E_m]) \tag{64}$$

There is a possibility of light splitting into $-\pi/2 < \beta < \pi/2$. With m_{min} as minimum integer grater than $d/\lambda \times (\sin\alpha - 1)$ and m_{max} as maximum integer

Band	$\lambda_{min}(\mu m)$	$\lambda_{max}(\mu m)$	allowed orders
J	1.13	1.37	-5 $< m <$ 5
H	1.50	1.80	-5 $< m <$ 5
K	2.01	2.42	-5 $< m <$ 5

Table 1: Allowed orders of diffraction with spectra not overlapping.

less than $d/\lambda \times (\sin\alpha + 1)$, total emergent intensity is $\sum_{m'=m_{min}}^{m_{max}} I_{m'}$. Thus, Efficiency regarding the order split is

$$\frac{I_m}{I_{total}} = \frac{\frac{d'}{d}\frac{2}{(kd'E_m)^2}(1-\cos[kd'E_m])}{\sum_{m'=m_{min}}^{m_{max}} \frac{d'}{d}\frac{2}{(kd'E_{m'})^2}(1-\cos[kd'E_{m'}])} \tag{65}$$

7 Free spectral range

From Equation 10, emergent angle depends on both λ and m when α and d are fixed. When taking spectra of a wavelength band from λ_{min} through λ_{max}, we have to be careful not to allow the light from the next orders $(m \pm 1)$ into the observed β. For this, following condition has to be satisfied:

$$\lambda_{max}(m-1) < \lambda_{min}m \text{ and } \lambda_{max}m < \lambda_{min}(m+1) \tag{66}$$

Table 1 shows the allowed m for the NIR bands when we want to take spectra of the whole band at once.

8 Model calculations of grating efficiency

Figures 4 – 9 shows the results from calculation in the Excel file for J-band and K-band gratings.

It turned out that:

- Lower order has higher efficiency.
- Better efficiency can be achieved when α and ε have the same sign.
- Incident angle and emergent angle should be as near as possible.
- Grating constant should be as big as possible.

All the results indicates that we get better efficiency with bigger optics.

A Derivation of α and β from Δ

Equation 12 is derived to satisfy the following conditions.

$$\begin{cases} \Delta = \alpha - \beta \\ \frac{m\lambda}{d} = \sin\alpha + \sin\beta \end{cases} \tag{67}$$

From $\beta = \alpha - \Delta$, $\sin\beta = \sin\alpha\cos\Delta - \cos\alpha\sin\Delta$.When we define $x \equiv \sin\alpha$ and $p \equiv \frac{m\lambda}{d}$,

$$\begin{aligned} \sin\beta &= \sin[\alpha - \Delta] && (68) \\ &= \sin\alpha\cos\Delta - \cos\alpha\sin\Delta && (69) \\ &= x\cos\Delta - \sqrt{1-x^2}\sin\Delta && (70) \end{aligned}$$

because $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$. Therefore, the second condition becomes

$$p = x + x\cos\Delta - \sqrt{1-x^2}\sin\Delta \tag{71}$$

and

$$\sqrt{1-x^2}\sin\Delta = x + x\cos\Delta - p \tag{72}$$

Taking square of the both sides,

$$(1-x^2)\sin^2\Delta = x^2(1+\cos\Delta)^2 - 2xp(1+\cos\Delta) + p^2 \tag{73}$$

or

$$0 = x^2[(1+\cos\Delta)^2 + \sin^2\Delta] - 2p(1+\cos\Delta)x + p^2 - \sin^2\Delta \tag{74}$$

When we define $x' \equiv \sin\beta$, an equation of the same shape comes out. This means that the two solutions $x_\pm$ of the second order equation corresponds to $\sin\alpha$ and $\sin\beta$.

Solutions of Equation 74 are

$$x_\pm = \frac{p(1+\cos\Delta) \pm \sin\Delta \times \sqrt{2(1+\cos\Delta) - p^2}}{2(1+\cos\Delta)} \tag{75}$$

Thus, α and β can be derived as $\arcsin x_\pm$.

Following is a perl script to calculate α and β.

```
#!/usr/bin/perl -w

# subroutines
# arcsin from man perlfunc
sub asin { atan2($_[0], sqrt(1 - $_[0] * $_[0])) }

# usage
$usage = << '_USAGE';
usage: alphabeta.pl delta(deg) m d lambda
  with delta=|alpha-beta|, m=diffraction order, lambda=wavelength
  prints out solutions for alpha(deg) and beta(deg)
_USAGE

# Main routine
# usage
if (@ARGV != 4) {
```

```
  print $usage;
  exit -1;
}

# get command line options
$delta = shift(@ARGV);
$m = shift(@ARGV);
$d = shift(@ARGV);
$lambda = shift(@ARGV);

# calculation
$pi = atan2(1,1)*4;
$cosd = cos($delta*$pi/180); # cos(delta)
$sind = sin($delta*$pi/180); # sin(delta)
$p = $m*$lambda/$d; # m*lambda/d
$p1 = (-$p*(1+$cosd)+$sind*sqrt(2*(1+$cosd)-$p*$p))/2/(1+$cosd);
$p2 = (-$p*(1+$cosd)-$sind*sqrt(2*(1+$cosd)-$p*$p))/2/(1+$cosd);
$a1 = asin($p1)*180/$pi; # solution in degrees
$a2 = asin($p2)*180/$pi; # solution in degrees

# results
print "$a1 $a2\n";
```

B File locations

Original source of this document is at `siroan:~mos/design/010521_Grating/` and copied to `gin-an:~tomono/Presen01/010521_Grating/`. Revised version is edited at `gin-an:~tomono/Presen01/011228_Grating/`. Some of the plots are generated from `siroan:/win98/My Documents/tomono/Design01/Fiber/010514_OptTrain/SlitFunctions.xls`. HTML version of this document is posted at `http://www.rzg.mpg.de/~tomono/MOS/auth/Reports/011228_Grating/`.

C Revision history

2001.5.23. -

- First release

2001.5.23. -

- Errata: parenthesis in Equation 4
- Another grating shape added (Figure 2(b), Equations 61, and 62)

2001.5.24. -

- Derivation of α and β from Δ

- Errata: $\frac{d'}{d}$ coefficient in Equation 65

2001.5.28. -

- J-band grating
- Plots are updated according to change of Equation
- Second release
- calculation of f_{cam}

2001.12.21. -

- Source file copied to tomono@gin-an
- Link to the postscript file added to the html version
- Spell checked

2001.12.28. - Revised version

- Consideration of wide entrance slit

2002.1.10. -

- Equations 12 and 75 are corrected.

2002.1.11. -

- Perl script added for calculation of α and β.

2002.1.17. -

- Section 3.4 added.

2002.2.4. -

- Section 4 added.

lmin	1.1					
lmax	1.4					
l_c(um)	1.25			1.25		
d(um)	5.83332			5.83332		
m	2			2		
delta(deg	-45			45		
alpha(deg	-9.0886			35.9114		
beta_c(de	35.9114			-9.0886		
epsilon(d	13.4114			13.4114		
N	5000			5000		
a(rad)	-0.1586			0.62677		
b_c(rad)	0.62677			-0.1586		
e(rad)	0.23407			0.23407		
l(um)	beta(deg)	Shadow	Total	beta(deg)	Shadow	Total
1.1	32.351	0.86878	0.66253	-12		
1.10476	32.4618	0.86829	0.67144	-11		
1.10952	32.5728	0.86781	0.68005	-11		
1.11429	32.6838	0.86732	0.68836	-		
1.11905	32.7951	0.86682	0.69644	-11		
1.12381	32.9064	0.86633	0.70434	-11		
1.12857	33.0179	0.86584	0.71268	-11		
1.13333	33.1295	0.86534	0.71976	-11		
1.1381	33.2413	0.86484	0.72656	-11		
1.14286	33.3532	0.86434	0.73311	-11		
1.14762	33.4653	0.86384	0.73943	-11		
1.15238	33.5775	0.86334	0.74553	-11		
1.15714	33.6898	0.86284	0.75144	-10		
1.1619	33.8023	0.86234	0.75719	-10		
1.16667	33.915	0.86183	0.76282	-10		
1.17143	34.0278	0.86133	0.76838	-10		
1.17619	34.1407	0.86082	0.77392	-1		
1.18095	34.2538	0.86031	0.77952	-10		
1.18571	34.3671	0.8598	0.78526	-1		
1.19048	34.4805	0.85928	0.79126	-10		
1.19524	34.594	0.85877	0.7976	-1		
1.2	34.7078	0.85826	0.80438	-10		
1.20476	34.8216	0.85774	0.8115	-9.9		
1.20952	34.9357	0.85722	0.81843	-9.8		
1.21429	35.0498	0.8567	0.82323	-9.7		
1.21905	35.1642	0.85618	0.81995	-9.7		
1.22381	35.2787	0.85566	0.79259	-		
1.22857	35.3934	0.85513	0.83047	-9.5		
1.23333	35.5082	0.8546	0.83246	-9.4		
1.2381	35.6232	0.85408	0.83425	-9.3		
1.24286	35.7384	0.85355	0.83583	-9.2		
1.24762	35.8537	0.85302	0.83721	-9		
1.25238	35.9692	0.85248	0.83839	-9.0		
1.25714	36.0849	0.85195	0.83937	-8.9		
1.2619	36.2007	0.85141	0.84015	-8.8		
1.26667	36.3167	0.85087	0.84072	-8.7		
1.27143	36.4329	0.85033	0.8411	-8.6		

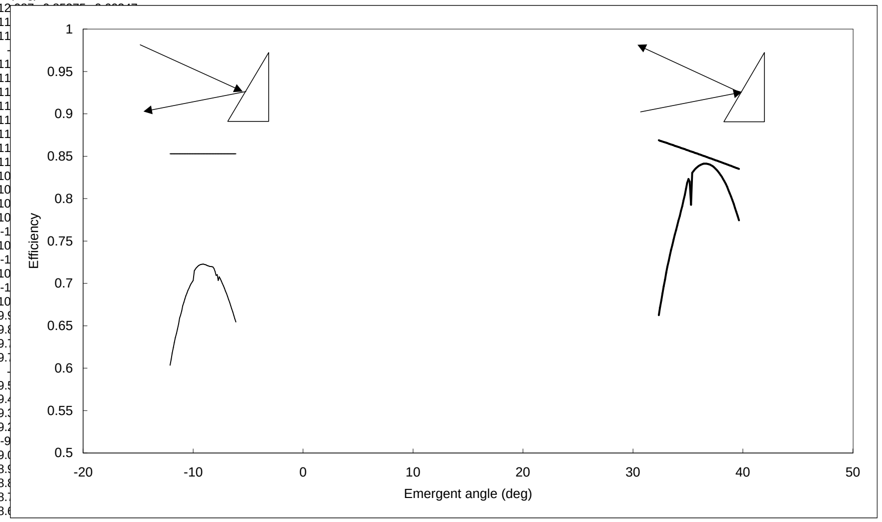


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Figure 4: Efficiency of the J-band grating plotted for emergent angle. d : $5.83\mu m$, ε : $13.4deg$ optimized for λ : $1.25\mu m$ with $\alpha - \beta = \pm 45deg$.

lmin	1.1					
lmax	1.4					
l_c(um)	1.25			1.25		
d(um)	5.83332			5.83332		
m	2			2		
delta(deg	-45			45		
alpha(deg	-9.0886			35.9114		
beta_c(de	35.9114			-9.0886		
epsilon(de	13.4114			13.4114		
N	5000			5000		
a(rad)	-0.1586			0.62677		
b_c(rad)	0.62677			-0.1586		
e(rad)	0.23407			0.23407		
l(um)	beta(deg)	Shadow	Total	beta(deg)	Shadow	Total
1.1	32.351	0.86878	0.66253	-12		
1.10476	32.4618	0.86829	0.67144	-11		
1.10952	32.5728	0.86781	0.68005	-11		
1.11429	32.6838	0.86732	0.68836	-		
1.11905	32.7951	0.86682	0.69644	-11		
1.12381	32.9064	0.86633	0.70434	-11		
1.12857	33.0179	0.86584	0.71268	-11		
1.13333	33.1295	0.86534	0.71976	-11		
1.1381	33.2413	0.86484	0.72656	-11		
1.14286	33.3532	0.86434	0.73311	-11		
1.14762	33.4653	0.86384	0.73943	-11		
1.15238	33.5775	0.86334	0.74553	-11		
1.15714	33.6898	0.86284	0.75144	-10		
1.1619	33.8023	0.86234	0.75719	-10		
1.16667	33.915	0.86183	0.76282	-10		
1.17143	34.0278	0.86133	0.76838	-10		
1.17619	34.1407	0.86082	0.77392	-1		
1.18095	34.2538	0.86031	0.77952	-10		
1.18571	34.3671	0.8598	0.78526	-1		
1.19048	34.4805	0.85928	0.79126	-10		
1.19524	34.594	0.85877	0.7976	-1		
1.2	34.7078	0.85826	0.80438	-10		
1.20476	34.8216	0.85774	0.8115	-9.9		
1.20952	34.9357	0.85722	0.81843	-9.8		
1.21429	35.0498	0.8567	0.82323	-9.7		
1.21905	35.1642	0.85618	0.81995	-9.7		
1.22381	35.2787	0.85566	0.79259	-		
1.22857	35.3934	0.85513	0.83047	-9.5		
1.23333	35.5082	0.8546	0.83246	-9.4		
1.2381	35.6232	0.85408	0.83425	-9.3		
1.24286	35.7384	0.85355	0.83583	-9.2		
1.24762	35.8537	0.85302	0.83721	-9		
1.25238	35.9692	0.85248	0.83839	-9.0		
1.25714	36.0849	0.85195	0.83937	-8.9		
1.2619	36.2007	0.85141	0.84015	-8.8		
1.26667	36.3167	0.85087	0.84072	-8.7		
1.27143	36.4329	0.85033	0.8411	-8.6		

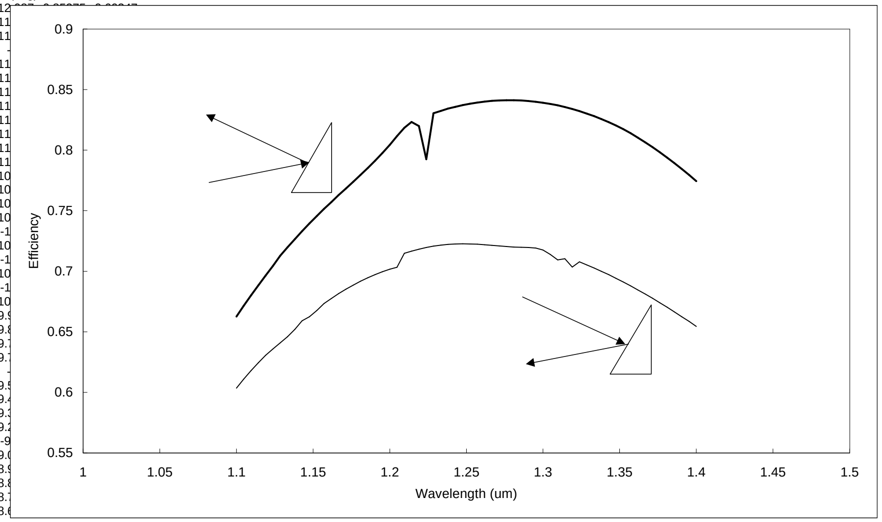

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Figure 5: Efficiency of the J-band grating plotted for wavelength. $d: 5.83\mu m$, $\varepsilon: 13.4deg$ optimized for $\lambda: 1.25\mu m$ with $\alpha-\beta=\pm 45deg$.

lmin	1.1					
lmax	1.4					
l_c(um)	1.25			1.25		
d(um)	5.83331875			5.83331875		
m	2			2		
delta(deg)	-45			45		
alpha(deg)	-9.08858139			35.91141861		
beta_c(deg)	35.91141861			-9.08858139		
epsilon(deg)	13.41141861			13.41141861		
N	5000			5000		
a(rad)	-0.15862567			0.626772494		
b_c(rad)	0.626772494			-0.15862567		
e(rad)	0.234073412			0.234073412		
m	beta(deg)	Shadow	Slit	beta(deg)	Shadow	Slit
-6						
-5.9						
-5.8						
-5.7						
-5.6						
-5.5						
-5.4	-87.6858732	1	0.0023219			
-5.3	-77.8925152	1	0.00192872			
-5.2	-73.0044028	1	0.00197603			
-5.1	-69.2117684	1	0.00213432			
-5	-65.9893685	1	0.00229918			
-4.9	-63.1308929	1	0.00239683			
-4.8	-60.5299197	1	0.00236933			
-4.7	-58.1230376	1	0.00218101			
-4.6	-55.8688562	1	0.0018289			
-4.5	-53.7385911	1	0.00135027			
-4.4	-51.7112658	1	0.0008224			
-4.3	-49.7710319	1	0.00035202			
-4.2	-47.9055663	1	5.4314E-05			
-4.1	-46.1050599	1	2.4726E-05			
-4	-44.3615504	1	0.00030988			
-3.9	-42.6684663	1	0.00088581			
-3.8	-41.0203066	1	0.00165151			
-3.7	-39.4124082	1	0.00244291			
-3.6	-37.8407751	1	0.00306764			
-3.5	-36.3019491	1	0.00335425			
-3.4	-34.7929113	1	0.00320411			
-3.3	-33.311005	1	0.0026307			
-3.2	-31.8538754	1	0.00177195			
-3.1	-30.4194211	1	0.00086691			
-3	-29.0057551	1	0.00019775			
-2.9	-27.6111732	1	9.3184E-06			
-2.8	-26.2341278	1	0.0004281			
-2.7	-24.873206	1	0.00140656			
-2.6	-23.5271119	1	0.00271521			
-2.5	-22.1946508	1	0.00399281			
-2.4	-20.8747166	1	0.00484758			
-2.3	-19.5662805	1	0.0049835			

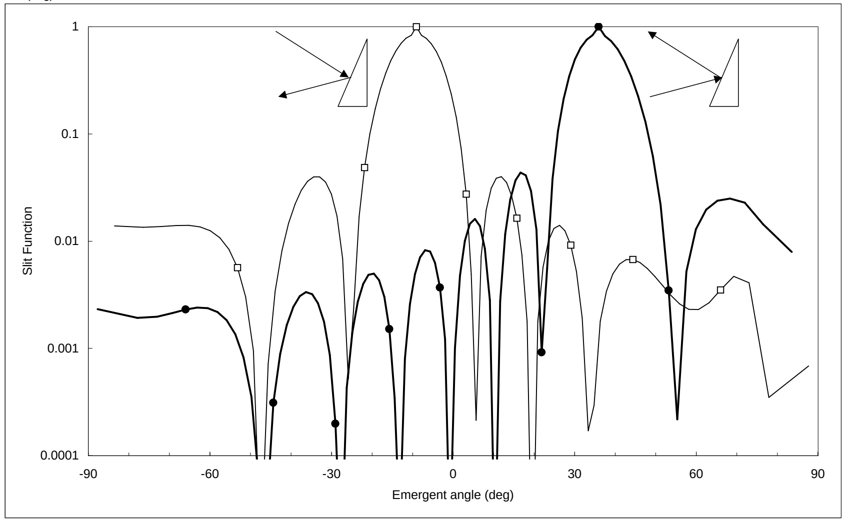

Figure 6: Slit function for different orders with $\alpha - \beta = +45deg$ and $-45deg$. Squares and circles indicates the integer orders which angles light of $1.25\mu m$ actually emitted. For the $\alpha - \beta = +45deg$ configuration, integer order exists in the wing of the primary peak of slit function, which makes the efficiency worth.

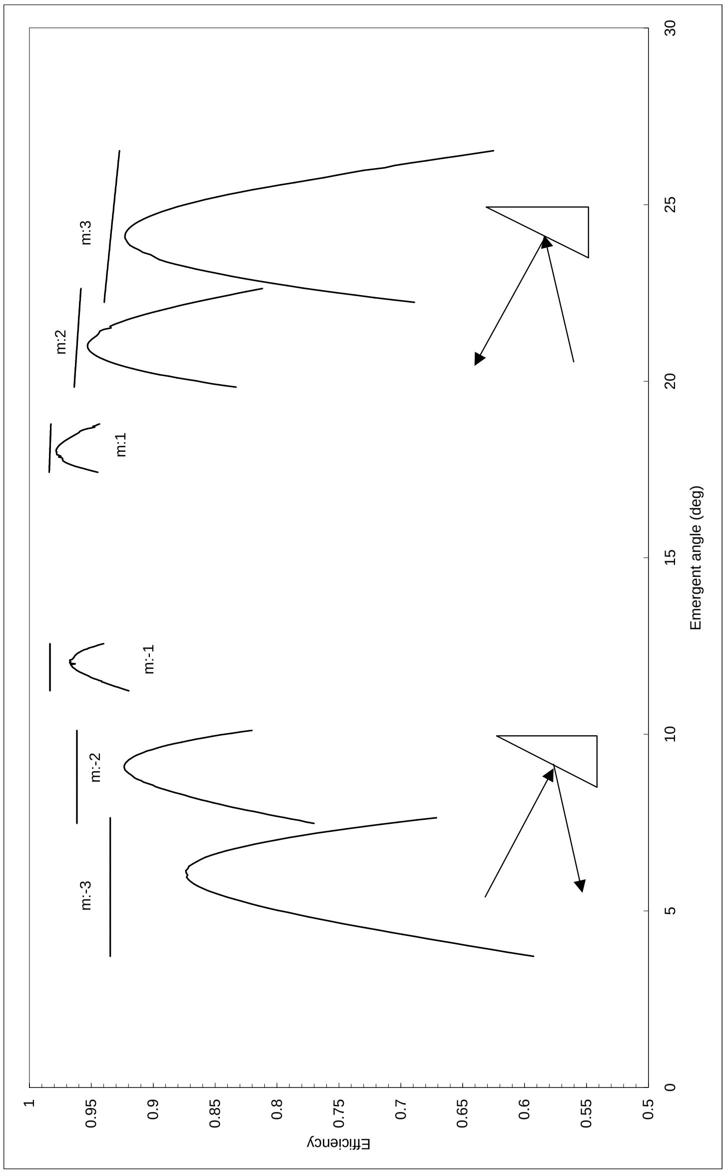

Figure 7: Change of efficiency in different orders for K-band grating. $d = 22\mu m$, $\Delta = -45deg$, $\lambda = 2 - -2.5\mu m$

lmin	2																	
lmax	2.5																	
l_c(um)	2.2			2.2			2.2			2.2			2.2			2.2		
d(um)	22			22			22			22			22			22		
m	1			1			1			1			1			1		
delta(deg	-60			-45			-30			30			45			60		
alpha(deg	-26.69			-19.398			-12.033			17.9672			25.6023			33.3098		
beta_c(de	33.3098			25.6023			17.9672			-12.033			-19.398			-26.69		
epsilon(d	3.30981			3.10234			2.96717			2.96717			3.10234			3.30981		
N	5000			5000			5000			5000			5000			5000		
a(rad)	-0.4658			-0.3386			-0.21			0.31359			0.44685			0.58137		
b_c(rad)	0.58137			0.44685			0.31359			-0.21			-0.3386			-0.4658		
e(rad)	0.05777			0.05415			0.05179			0.05179			0.05415			0.05777		
l(um)	beta(deg)	Shadow	Total	beta(deg)	Shadow	Total	beta(deg)	Shadow	Total	beta(deg)	Shadow	Total	beta(deg)	Shadow	Total	beta(deg)	Shadow	Total
2	32.6887	0.96422	0.91817	25.0261	0.97532	0.93546	17.4204	0.984	0.94471	-12.566	0.98347	0.94006	-19.951	0.97469	0.91718	-27.275	0.96339	0.88823
2.00794	32.7133	0.96418	0.92															
2.01587	32.7379	0.96415	0.92															
2.02381	32.7625	0.96412	0.93															
2.03175	32.787	0.96409	0.93															
2.03968	32.8116	0.96405	0.93															
2.04762	32.8362	0.96402	0.93															
2.05556	32.8608	0.96399	0.93															
2.06349	32.8854	0.96396	0.94															
2.07143	32.9101	0.96392	0.94															
2.07937	32.9347	0.96389	0.94															
2.0873	32.9593	0.96386	0.94															
2.09524	32.9839	0.96382	0.9															
2.10317	33.0086	0.96379	0.94															
2.11111	33.0332	0.96376	0.94															
2.11905	33.0579	0.96373	0.94															
2.12698	33.0826	0.96369	0.95															
2.13492	33.1072	0.96366	0.95															
2.14286	33.1319	0.96363	0.95															
2.15079	33.1566	0.96359	0.95															
2.15873	33.1813	0.96356	0.95															
2.16667	33.206	0.96353	0.95															
2.1746	33.2307	0.96349	0.95															
2.18254	33.2554	0.96346	0.95															
2.19048	33.2801	0.96343	0.95															
2.19841	33.3049	0.9634	0.95															
2.20635	33.3296	0.96336	0.95															
2.21429	33.3543	0.96333	0.95															
2.22222	33.3791	0.9633	0.95															
2.23016	33.4038	0.96326	0.95															
2.2381	33.4286	0.96323	0.95															
2.24603	33.4534	0.9632	0.95															
2.25397	33.4782	0.96316	0.9															
2.2619	33.5029	0.96313	0.95															
2.26984	33.5277	0.9631	0.95															
2.27778	33.5525	0.96306	0.95															
2.28571	33.5773	0.96303	0.95															

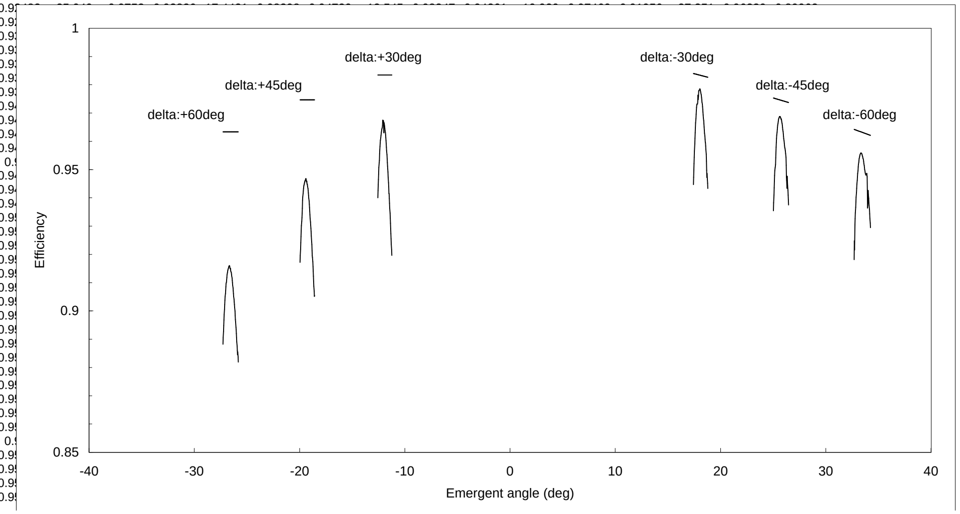

Figure 8: Change of efficiency in different $\Delta = \alpha - \beta$ for the K-band grating. $d = 22\mu m$, $m = 1$.

lmin	2																	
lmax	2.5																	
l_c(um)	2.2			2.2			2.2			2.2			2.2			2.2		
d(um)	22			44			88			176			352			704		
m	1			1			1			1			1			1		
delta(deg	-30			-30			-30			-30			-30			-30		
alpha(deg	-12.033			-13.517			-14.259			-14.629			-14.815			-14.907		
beta_c(de	17.9672			16.4831			15.7415			15.3707			15.1854			15.0927		
epsilon(de	2.96717			1.48309			0.74148			0.37073			0.18537			0.09268		
N	5000			5000			5000			5000			5000			5000		
a(rad)	-0.21			-0.2359			-0.2489			-0.2553			-0.2586			-0.2602		
b_c(rad)	0.31359			0.28768			0.27474			0.26827			0.26503			0.26342		
e(rad)	0.05179			0.02588			0.01294			0.00647			0.00324			0.00162		
l(um)	beta(deg)	Shadow	Total	beta(deg)	Shadow	Total	beta(deg)	Shadow	Total	beta(deg)	Shadow	Total	beta(deg)	Shadow	Total	beta(deg)	Shadow	Total
2	17.4204	0.984	0.94471	16.2117	0.99253	0.95681	15.6062	0.9964	0.96221	15.3032	0.99823	0.96489	15.1516	0.99912	0.96622	15.0758	0.99956	0.96688
2.00794	17.4421	0.98398	0.94															
2.01587	17.4638	0.98395	0.94															
2.02381	17.4854	0.98393	0.95															
2.03175	17.5071	0.98391	0.95															
2.03968	17.5288	0.98389	0.95															
2.04762	17.5505	0.98387	0.95															
2.05556	17.5721	0.98385	0.96															
2.06349	17.5938	0.98383	0.96															
2.07143	17.6155	0.98381	0.96															
2.07937	17.6372	0.98379	0.96															
2.0873	17.6589	0.98377	0.96															
2.09524	17.6806	0.98375	0.96															
2.10317	17.7023	0.98372	0.9															
2.11111	17.724	0.9837	0.97															
2.11905	17.7457	0.98368	0.97															
2.12698	17.7674	0.98366	0.9															
2.13492	17.7891	0.98364	0.97															
2.14286	17.8108	0.98362	0.97															
2.15079	17.8325	0.9836	0.97															
2.15873	17.8542	0.98358	0.97															
2.16667	17.8759	0.98356	0.97															
2.1746	17.8977	0.98354	0.97															
2.18254	17.9194	0.98352	0.97															
2.19048	17.9411	0.98349	0.97															
2.19841	17.9628	0.98347	0.97															
2.20635	17.9846	0.98345	0.97															
2.21429	18.0063	0.98343	0.97															
2.22222	18.028	0.98341	0.97															
2.23016	18.0498	0.98339	0.97															
2.2381	18.0715	0.98337	0.97															
2.24603	18.0932	0.98335	0.97															
2.25397	18.115	0.98333	0.97															
2.2619	18.1367	0.9833	0.97															
2.26984	18.1585	0.98328	0.97															
2.27778	18.1802	0.98326	0.97															
2.28571	18.202	0.98324	0.97															

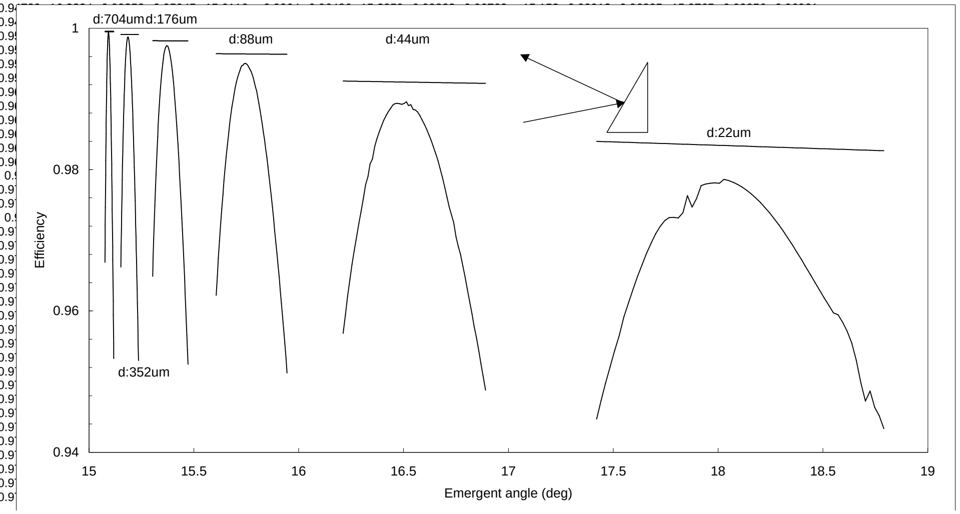

Figure 9: Efficiency dependence on groove interval d for the K-band grating. $\Delta = -30deg$, $m = 1$.